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Developing flow on a vertical wall

By K. J. RUSCHAK† AND L. E. SCRIVEN

Department of Chemical Engineering and Materials Science,
University of Minnesota, Minneapolis

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Developing laminar flow in a liquid film issuing from a full slot and descending along a vertical wall is analysed for the limiting cases of low flow rate and high surface tension. Neither limiting case is approached uniformly and the singular perturbation method with matched asymptotic expansions is employed. The location of a free boundary, the curved meniscus, is unknown *a priori*, and the limiting meniscus profiles are obtained. An interesting viscous boundary layer is found at low flow rates.

1. Introduction

This paper reports one step towards understanding the fluid mechanics of free-surface flows. We analyse two limiting cases of the transition flow of liquid along a vertical wall from a full slot to a falling film of uniform thickness. The geometric configuration considered has several applications. It may be encountered, for example, at the top of a wetted wall column. It may also be adapted as a prototype model for the distribution slot of a slide hopper, which is an important industrial coating tool (Mercier *et al.* 1956).

We first identify two regions of the flow field in which different combinations of pressure, viscous tractions, body force and boundary conditions dominate as the ratio of the downstream film thickness to the slot width assumes small values. We then guess the nature of a principal limit of the problem and introduce appropriately scaled dimensionless variables, and finally we carry out asymptotic analyses, matching in the usual manner the representations of the flow field obtained in the two regions. We treat in the same way a second limiting case, in which the ratio of slot width to the capillary length is small.

The results provide insight into the interplay of factors which influence the entire range of flow conditions, not just the limiting cases which can be handled by asymptotic methods. They illuminate viscous flow bounded by a curved meniscus whose location is unknown *a priori*: an important class of free-boundary problems which is still not well understood. Moreover the analysis is of interest as a singular perturbation problem in which a viscous boundary layer emerges in the limit of small flow rate. Computer simulations of viscous flows with curved free boundaries are being developed. The asymptotic solutions provide test cases for numerical methods and first approximations for numerical solution of the governing equations, as well as answers when the length scales in the simulation are so disparate that numerical methods are impracticable.

† Present address: Research Laboratories, Eastman Kodak Company, Rochester, New York.

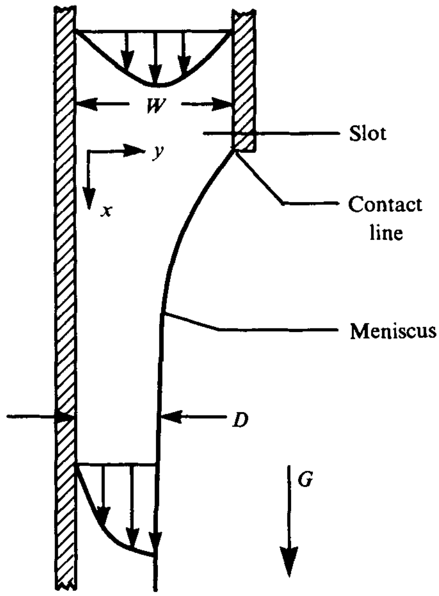

FIGURE 1. Developing flow from a slot.

Previous studies of developing flow from a slot have used the boundary-layer equation and have neglected surface tension. Cerro & Whitaker (1971) solve the boundary-layer equation numerically. Stucheli & Ozisik (1976) and Murty & Sastri (1973) integrate the boundary-layer equation across the film and substitute a guessed velocity profile in order to reduce the problem to an ordinary differential equation for the film thickness. Thomas & Rice (1973) report measured velocity profiles. The two limiting cases studied here have not been previously examined to our knowledge.

Developing flow of a falling liquid film is considered in the limit of low flow rate in § 3 and in the limit of high surface tension in § 4. In both cases the principal limit of the problem's solution is approached non-uniformly, and the singular perturbation method with matched asymptotic expansions is used to begin an asymptotic representation of the flow field. The scaling of the low-flow-rate limiting case is of particular interest, because it involves disparate length scales in both spatial dimensions.

2. Mathematical formulation

Newtonian liquid of uniform properties flows through a slot comprised of two vertical walls (figure 1). One wall of the slot terminates, and below this point there is a liquid/air interface. The slot is sufficiently long that Poiseuille flow is established well upstream of the end of the slot, and the continuing wall is long enough that a gravity-driven, rectilinear flow develops downstream. The ambient air exerts negligible viscous traction on the interface and, being much less dense than the liquid, is modelled by a uniform pressure. The developing film flow is steady and two-dimensional.

The liquid properties are the density ρ, kinematic viscosity ν and surface tension σ. The acceleration due to gravity is G, the slot width is W and the downstream film thickness is D. A convenient derived length is the capillary length $C = (\sigma/\rho G)^{\frac{1}{2}}$ characteristic of free static menisci. An x, y Cartesian co-ordinate system is oriented as shown in figure 1.

We first scale the governing equations for the region near the end of the slot. Here the horizontal dimension of the film is of the order of the slot width, and distance in this region is measured in units of W. The magnitude of the liquid velocity is of the order of that associated with the Poiseuille flow, and consequently the velocity is made dimensionless with $GD^3/\nu W$. The first limiting case considered below is that of low flow rate, and anticipating nearly hydrostatic conditions near the slot exit when the flow rate is sufficiently small, we scale the difference between the pressure in the liquid and atmospheric pressure with ρGW. The dimensionless governing equations are then

$$\left.\begin{aligned} R[uu_x+vu_y] &= -d^{-3}p_x+u_{yy}+u_{xx}+d^{-3}, \\ R[uv_x+vv_y] &= -d^{-3}p_y+v_{yy}+v_{xx}, \\ u_x+v_y &= 0. \end{aligned}\right\} \tag{2.1}$$

On the rigid boundaries $u = 0$ and $v = 0$. The free-surface boundary conditions on $y = h(x)$, $x > 0$, are

$$\left.\begin{aligned} -p &= d^3[2u_x+h_x(u_y+v_x)]+w^{-2}h_{xx}[1+h_x^2]^{-\frac{3}{2}}, \\ 0 &= [u_y+v_x][1-h_x^2]-4h_xu_x, \quad v = h_xu. \end{aligned}\right\} \tag{2.2}$$

The asymptotic regimes upstream and downstream, respectively, are

$$u \to 2[y-y^2], \quad v\to 0 \quad \text{as} \quad x\to -\infty, \quad 0 < y < 1, \tag{2.3}$$

$$u \to d^{-3}[dy-\tfrac{1}{2}y^2], \quad v\to 0, \quad h\to d \quad \text{as} \quad x\to\infty, \quad 0 < y < d. \tag{2.4}$$

Here u and v are the x and y components of the dimensionless velocity vector, p is the dimensionless pressure, and $h(x)$ is the value of y at the free surface. The three dimensionless parameters are $d = D/W$, the ratio of the film thickness far downstream to the width of the slot; $w = W/C$, the ratio of the slot width to the capillary length; and $R = GD^3/\nu^2 = (GD^3/\nu W)\, W/\nu$, the Reynolds number.

That the boundary conditions given here model the flow realistically near the three-phase line where the liquid/air meniscus contacts the solid is not certain. Michael (1958) and Richardson (1967) have shown that when inertial effects are neglected, and when the meniscus and rigid wall are planar and unbounded, the above equations demand that the meniscus and rigid wall be coplanar and predict unbounded velocity gradients and a negatively unbounded pressure at the contact line. Casual observations suggest that angles between free and rigid surfaces other than π can occur where liquid flows near a stationary contact line. It is not clear, however, that the work of Michael and Richardson represents a local analysis of (2.1)–(2.4). We do not carry our analyses to the point where the flow near the contact line must be examined. We detail the nearly rectilinear, downstream portion of the flow field, which, in the two limiting situations we consider, is unaffected by the details of flow in the slot, and indeed of the exact configuration of the slot.

Equations (2.1) are too complex to solve analytically. Asymptotic analysis is in order.

3. Limiting case of small flow rate

When $d = D/W$ is small there are two horizontal length scales. Near the slot exit the liquid film has a thickness of the order of W, but downstream the thickness is of the order of $D \ll W$. If viscous, pressure and surface-tension forces are of equal importance where the film has nearly its final thickness D, then viscous forces must be of secondary importance near the slot exit. That is, surface tension and a pressure field which is nearly hydrostatic should dominate the shape of the free surface just downstream of the slot.

When the formal limit of (2.1)–(2.4) as $d \to 0$ is taken, the following simplified equations result:

$$\left.\begin{aligned} 0 = -p^0_x + 1, \quad 0 = -p^0_y, \\ -p^0(x, h^0) = w^{-2} h^0_{xx} [1 + (h^0_x)^2]^{-\frac{3}{2}}, \\ h^0 = 1 \quad \text{at} \quad x = 0. \end{aligned}\right\} \tag{3.1}$$

A superscript zero denotes the limit of a function as $d \to 0$. Equations (3.1) are equivalent to the Laplace–Young equation for static menisci, and the limiting profile is an elastica determined entirely by a hydrostatic pressure field and surface tension (Maxwell & Rayleigh 1910). Two additional boundary conditions are needed before the static meniscus which is the limiting free surface near the slot exit can be specified.

The dependent and independent variables must be redefined to reflect the change in length scales in the region of viscous flow downstream. To motivate our second set of variables we briefly outline a possible scaling procedure. Clearly y and h can be divided by d to obtain replacement variables of order unity. Moreover, the rectilinear flow far downstream establishes that u is of magnitude $1/d$ there. The new x co-ordinate must reflect the vertical length over which viscous forces are important as well as the distance below the slot at which viscous forces become important. We therefore anticipate the form $(x-l)/d^n$ for the rescaled x co-ordinate, where $l > 0$ and $n > 0$ are yet to be determined. When the two terms in the continuity equation are required to be of the same order, v is found to be of order d^{-n}. To rescale the pressure variable and determine the value of n, we can anticipate that pressure, viscous and surface-tension forces are equally important where the film thickness is nearly D. Alternatively we can pursue the assumption that the upstream and downstream expressions for the pressure and surface profile match in intermediate limits (Van Dyke 1964, p. 91) to complete the rescaling.

The variables scaled for the region of viscous flow are

$$\left.\begin{aligned} \bar{x} \equiv (x-l)/d^{\frac{1}{3}}, \quad \bar{y} \equiv y/d, \quad \bar{h} \equiv h/d, \\ \bar{u} \equiv du, \quad \bar{v} \equiv d^{\frac{1}{3}} v, \quad \bar{p} \equiv p/d^{\frac{1}{3}}. \end{aligned}\right\} \tag{3.2}$$

Only l, the value of x near which viscous forces become important, is unspecified. The length above $x = l$ over which viscous forces are significant is found to be of order $d^{\frac{1}{3}}$ and is much larger than d, the magnitude of the film thickness downstream. The slope of the film profile in the viscous region is small, being of order $d^{\frac{2}{3}}$, and the horizontal component of velocity v is smaller than the vertical component u by a factor of order $d^{\frac{2}{3}}$. In terms of the variables (3.2) the problem is

$$R d^{\frac{2}{3}} [\bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}}] = -\bar{p}_{\bar{x}} + \bar{u}_{\bar{y}\bar{y}} + d^{\frac{4}{3}} \bar{u}_{\bar{x}\bar{x}} + 1, \tag{3.3a}$$

$$Rd^2[\overline{u}\overline{v}_{\bar{x}} + \overline{v}\overline{v}_{\bar{y}}] = -\overline{p}_{\bar{y}} + d^{\frac{4}{3}}\,\overline{v}_{\bar{y}\bar{y}} + d^{\frac{8}{3}}\,\overline{v}_{\bar{x}\bar{x}}, \tag{3.3b}$$

$$\overline{u}_{\bar{x}} + \overline{v}_{\bar{y}} = 0, \tag{3.3c}$$

$$\left.\begin{aligned} -\overline{p} &= d^{\frac{4}{3}}[2\overline{u}_{\bar{x}} + \overline{h}_{\bar{x}}(\overline{u}_{\bar{y}} + d^{\frac{4}{3}}\overline{v}_{\bar{x}})] + w^{-2}\overline{h}_{\bar{x}\bar{x}}[1 + d^{\frac{4}{3}}\overline{h}^2_{\bar{x}}]^{-\frac{3}{2}} & (3.3d)\\ 0 &= (\overline{u}_{\bar{y}} + d^{\frac{4}{3}}\overline{v}_{\bar{x}})(1 - d^{\frac{4}{3}}\overline{h}^2_{\bar{x}}) - 4d^{\frac{4}{3}}\overline{h}_{\bar{x}}\,\overline{u}_{\bar{x}} & (3.3e)\\ \overline{v} &= \overline{h}_{\bar{x}}\,\overline{u} & (3.3f) \end{aligned}\right\} \quad \text{on} \quad \overline{y} = \overline{h}(\overline{x})$$

$$\overline{u} = \overline{v} = 0 \quad \text{on} \quad \overline{y} = 0, \tag{3.3g}$$

$$\overline{u} \to \overline{y} - \tfrac{1}{2}\overline{y}^2, \quad \overline{v} \to 0, \quad \overline{h} \to 1 \quad \text{as} \quad \overline{x} \to \infty, \quad 0 < \overline{y} < 1. \tag{3.3h}$$

The equations which result when d is set to zero in (3.3) are

$$0 = -\overline{p}^0_{\bar{x}} + \overline{u}^0_{\bar{y}\bar{y}} + 1, \quad 0 = -\overline{p}^0_{\bar{y}}, \quad \overline{u}^0_{\bar{x}} + \overline{v}^0_{\bar{y}} = 0, \tag{3.4a–c}$$

$$-\overline{p}^0(\overline{x}, \overline{h}^0) = w^{-2}\overline{h}^0_{\bar{x}\bar{x}}, \quad \overline{u}^0_{\bar{y}}(\overline{x}, \overline{h}^0) = 0, \quad \overline{v}^0(\overline{x}, \overline{h}^0) = \overline{h}^0_{\bar{x}}\,\overline{u}^0(\overline{x}, \overline{h}^0), \tag{3.4d–f}$$

$$\overline{u}^0 = \overline{v}^0 = 0 \quad \text{on} \quad \overline{y} = 0, \tag{3.4g}$$

$$\overline{u}^0 \to \overline{y} - \tfrac{1}{2}\overline{y}^2, \quad \overline{v}^0 \to 0, \quad \overline{h}^0 \to 1 \quad \text{as} \quad \overline{x} \to \infty, \quad 0 < y < 1. \tag{3.4h}$$

Note that the limiting pressure varies vertically but is constant across the film. Indeed the pressure depends on the profile curvature, and the resulting pressure gradient, along with gravity, drives the nearly rectilinear flow in the film. Equations (3.4) can be developed to give

$$\overline{p}^0_{\bar{x}} = 1 - (\overline{h}^0)^{-3}, \tag{3.5a}$$

$$\overline{u}^0 = (\overline{h}^0)^{-1}[\overline{y}/\overline{h}^0 - \tfrac{1}{2}(\overline{y}/\overline{h}^0)^2], \quad \overline{v}^0 = \overline{h}^0_{\bar{x}}(\overline{y}/\overline{h}^0)\,\overline{u}^0, \tag{3.5b, c}$$

$$\overline{h}^0_{\bar{x}\bar{x}\bar{x}} = w^2[(\overline{h}^0)^{-3} - 1]; \quad \overline{h}^0 \to 1 \quad \text{as} \quad \overline{x} \to \infty. \tag{3.5d, e}$$

The velocity profile is semi-parabolic and the streamline inclination is proportional to the slope of the free surface.

Let $g(r)$ be the solution to

$$g_{rrr} = g^{-3} - 1; \quad g \sim 1 + 10^{-4}\exp[-3^{\frac{1}{3}}(r-5)] \quad \text{as} \quad r \to \infty \tag{3.6}$$

(the numbers 10^{-4} and 5 were chosen arbitrarily). The solution to the third-order differential equation (3.5d) for $\overline{h}^0$ is then

$$\overline{h}^0(\overline{x}) = g(w^{\frac{2}{3}}\overline{x} + b) \tag{3.7}$$

for some value of b. The effect of a change in the value of b is to translate the profile without changing its shape. The asymptotic behaviour of g as $r \to -\infty$ is given by

$$g \sim -\tfrac{1}{6}r^3 + \tfrac{1}{2}c_0 r^2 + c_1 r + c_2 + \tfrac{9}{14}r^{-6} + \ldots, \tag{3.8}$$

where the c_i are known constants. We have determined g numerically, and the results are given in figure 2. Of course $\overline{h}^0$ could be expressed in terms of any translation of g parallel to the r axis, and any function obtained in this way would have an asymptotic expansion of the form (3.8). Although the values of the c_i change with translation, the values of

$$I_1 = \tfrac{1}{2}c_0^2 + c_1, \quad I_2 = \tfrac{1}{3}c_0^3 + c_0 c_1 + c_2 \tag{3.9}$$

depend only on the shape of the function and remain unchanged.

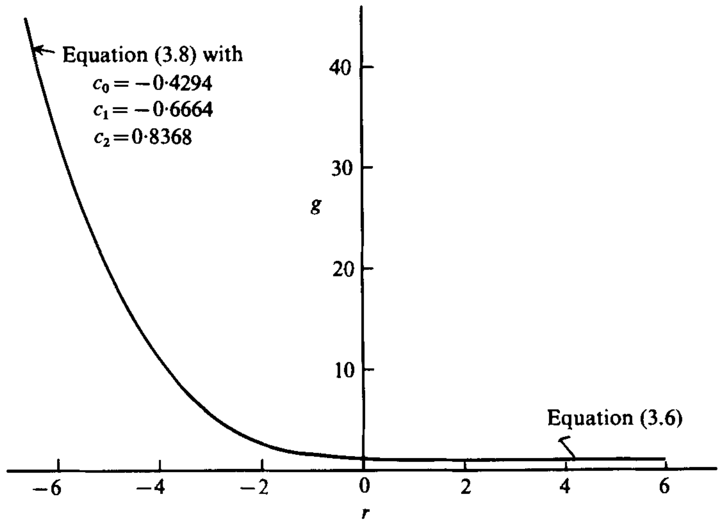

FIGURE 2. The meniscus-profile function for the viscous region $g(r)$. See (3.6), (3.8) and (3.9).

The profile (3.7) obtained for the region of viscous flow must be matched with $h^0(x)$, the static meniscus found near the slot exit. For this purpose the power-series expansion of $h^0(x)$ about $x = l$ is required; it has the form

$$h^0(x) = \sum_{n=0}^{\infty} e_n(x-l)^n. \tag{3.10}$$

Matching in the usual manner (Van Dyke 1964) we find that

$$e_0 = e_1 = e_2 = 0, \quad e_3 = -\tfrac{1}{6}w^2. \tag{3.11a, b}$$

As previously remarked, two additional boundary conditions are needed before (3.1) can be solved for $h^0(x)$. The relations (3.11) are restrictions on h^0 at $x = l$, where l is not yet known. Consequently three conditions on h^0 at $x = l$ are required to determine both h^0 and l; from (3.11)

$$h^0(l) = h^0_x(l) = h^0_{xx}(l) = 0. \tag{3.12}$$

Though no more degrees of freedom are available, it turns out that (3.11 b) is satisfied; see (3.19).

The solution to the combined problem (3.12) and (3.1) can be expressed in terms of elliptic integrals:

$$p^0 = x - l, \tag{3.13}$$

$$h^0 = w^{-1}\{2[E - E(k,\phi)] - [K - F(k,\phi)]\}, \tag{3.14}$$

$$w = 2[E - E(k,\phi_0)] - [K - F(k,\phi_0)], \tag{3.15}$$

$$\phi(x) \equiv \sin^{-1}[1 - \tfrac{1}{2}w^2(l-x)^2]^{\frac{1}{2}}, \tag{3.16}$$

$$\phi_0 \equiv \phi(0) = \sin^{-1}[1 - \tfrac{1}{2}w^2l^2]^{\frac{1}{2}}. \tag{3.17}$$

Here $F(k,\phi)$ and $E(k,\phi)$ are the elliptic integrals of the first and second kind, respectively. K is the complete elliptic integral of the first kind and E is the complete elliptic integral of the second kind. The modulus k is $1/\sqrt{2}$. Equation (3.13) gives the hydrostatic pressure field and (3.14) the static meniscus obtained in the limit of small flow rate. Equation (3.15) gives w as a function of wl. We have restricted ourselves to the case $0 < wl < \sqrt{2}$ for the convenience that $h^0(x)$ be single valued, although solutions exist outside this interval.

When the capillary length is much larger than the slot width, w is small and

$$\left.\begin{aligned} h^0 &\sim \tfrac{1}{6}(6^{\frac{1}{3}} - w^{\frac{2}{3}}x)^3 - \tfrac{3}{56}\,6^{\frac{2}{3}}w^{\frac{4}{3}}(6^{\frac{1}{3}} - w^{\frac{2}{3}}x)^2 \\ &\quad + \tfrac{1}{112}w^{\frac{4}{3}}(6^{\frac{1}{3}} - w^{\frac{2}{3}}x)^7 + \ldots \\ l &\sim 6^{\frac{1}{3}}w^{-\frac{2}{3}} - \tfrac{1}{56}\,6^{\frac{5}{3}}w^{\frac{2}{3}} + \ldots \end{aligned}\right\} \quad \text{as} \quad w \to 0. \tag{3.18}$$

The power-series expansion of h^0 about $x = l$ begins

$$h^0(x) \sim \tfrac{1}{6}w^2(l-x)^3 + \tfrac{1}{112}w^6(l-x)^7 + \ldots, \tag{3.19}$$

and consequently the relation (3.11*b*) demanded by matching is satisfied.

The remaining unknown is the constant b appearing in (3.7); it must be determined by matching at higher orders. If the asymptotic expansions valid near the slot exit proceed

$$\left.\begin{aligned} h &\sim h^0 + d^{\frac{1}{3}}h^{(1)} + \ldots, \\ p &\sim p^0 + d^{\frac{1}{3}}p^{(1)} + \ldots, \end{aligned}\right\} \tag{3.20}$$

then viscous forces play no part in determining $h^{(1)}$ and $p^{(1)}$, and the perturbation is onto another static meniscus. The requirements of matching and the fact that

$$h^{(1)}(0) = 0$$

can be used to show that $h^{(1)}$ and $p^{(1)}$ are both zero. The fact that $p^{(1)}$ is zero implies, again through matching, that $b = c_0$. As a result the final expression for $\bar{h}^0$ is

$$\bar{h}^0(\bar{x}) = g(w^{\frac{2}{3}}\bar{x} + c_0), \quad c_0 = -0{\cdot}4294. \tag{3.21}$$

Equations (3.14) and (3.21) give the entire meniscus profile when the flow rate is small, which implies $D \ll W$. In this case the film thickness varies widely, and viscous forces are important only in the narrower portions, which are downstream. Near the slot the film is relatively thick and the effects of liquid motion are small. Pressure, surface-tension and gravitational forces dominate, and the meniscus has a static equilibrium shape. Near $x = l$ the slope and curvature of the meniscus become small and the film thin enough that viscous forces begin to be important, influencing meniscus shape through the pressure at the free boundary. Further downstream, as pressure forces wane, the viscous forces come into balance with the force of gravity, and the meniscus approaches planarity. Throughout the region of viscous effects, the velocity and pressure distributions can be found from the meniscus profile, according to (3.5).

The limit of the film profile as $d \to 0$ with position with respect to the slot fixed (x fixed) is the static meniscus $h^0(x)$ for $0 < x \leqslant l$ and the line $y = 0$ for $x > l$, which might be thought of as a film of zero thickness. The length l, given by (3.15), or the asymptotic approximation (3.18), is the distance below the slot exit at which the elastica of which the static meniscus is a segment contacts the wall. According to (3.12) the elastica is tangential and has an inflexion point there: an osculating contact.

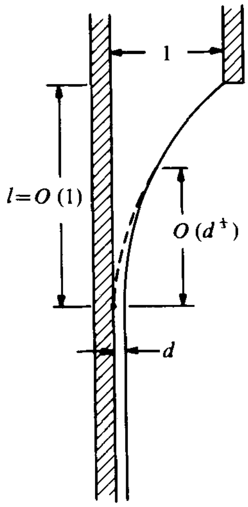

FIGURE 3. Film profile when the flow rate is low. The broken line is part of the limiting static meniscus which begins at the contact line. The film profile departs from this limiting static meniscus over the viscous transition region which leads to the rectilinear flow downstream.

Consequently this limit of the film profile has continuous curvature, but higher derivatives are discontinuous at $x = l$.

It is the viscous flow, lost in the principal limit, which smooths the transition to a planar meniscus downstream. This viscous transition region collapses towards the point $x = l$ on the wall as the flow rate decreases. Its structure is retained in a second limit process detailed in (3.2) in which the co-ordinates x and y move towards the point $x = l$ on the wall to maintain a relative position within the collapsing viscous transition region. The situation is summarized in figure 3.

When d is sufficiently large the shape of the free surface near the slot is significantly influenced by dynamic forces. The equations which predict the first effects of viscosity when the Reynolds number is small are, from (2.1)–(2.3),

$$0 = -p_x^{(i)} + u_{yy}^0 + u_{xx}^0, \qquad (3.22a)$$

$$0 = -p_y^{(i)} + v_{yy}^0 + v_{xx}^0, \qquad (3.22b)$$

$$0 = u_x^0 + v_y^0, \qquad (3.22c)$$

$$\left.\begin{aligned} 0 &= [u_y^0 + v_x^0][1-(h_x^0)^2] - 4h_x^0 u_x^0 \\ v^0 &= h_x^0 u^0 \end{aligned}\right\} \quad \text{on} \quad y = h^0(x), \qquad (3.22d),\ (3.22e)$$

$$u^0 \to 2[y-y^2], \quad v^0 \to 0 \quad \text{as} \quad x \to -\infty, \quad 0 < y < 1, \qquad (3.22f)$$

$$u^0 = v^0 = 0 \quad \text{on} \quad y = 0; y = 1, x < 0. \qquad (3.22g)$$

This is a difficult slow-flow problem, even though the domain on which the field equations are to be solved is specified. The normal-stress boundary conditions (2.2) generate a second-order differential equation for the perturbation of the surface profile caused by this flow. Joseph (1974) and Joseph & Sturges (1975) have encountered and solved problems of this type in more regular domains. Joseph does not discuss the nature of his solutions near the contact line.

4. Limiting case of large surface tension

A second limiting case of some interest is $w = W/C \to 0$. Because surface tension appears only in this parameter, the limit may be thought of as surface tension tending to infinity. It is reasonable to presume that at a fixed station $x > 0$ the limit of the film thickness h is 1, because the effect of increasing surface tension ought to be decreasing meniscus curvature. Indeed the curvature ought to become uniformly small as $\sigma \to \infty$, and if the meniscus is everywhere no more than slightly curved the flow beneath it should be nearly one-dimensional, except in the immediate vicinity of the slot exit. There, owing to the change from the no-slip boundary condition to the free-surface boundary condition, the flow is certainly two-dimensional. Finally, because $h \to d$ as $x \to \infty$ regardless of the value of w, it is clear that the limit $h(x) \to 1$, $w \to 0$ is not approached uniformly in x. We conclude that the singular perturbation method is once again required.

We postulate that asymptotic expansions in w begin

$$\left.\begin{aligned} p &\sim Aw^{-a} + \hat{p}^0 + \dots \\ u &\sim \hat{u}^0 + \dots, \quad v \sim \hat{v}^0 + \dots \\ h &\sim 1 + \dots \end{aligned}\right\} \quad \text{as} \quad w \to 0, \tag{4.1}$$

where a and A are constants, and we introduce these expressions into (2.1)–(2.4). The first non-trivial problem is

$$R[\hat{u}^0\hat{u}^0_x + \hat{v}^0\hat{u}^0_y] = -d^{-3}\hat{p}^0_x + \hat{u}^0_{yy} + \hat{u}^0_{xx} + d^{-3}, \tag{4.2a}$$

$$R[\hat{u}^0\hat{v}^0_x + \hat{v}^0\hat{v}^0_y] = -d^{-3}\hat{p}^0_y + \hat{v}^0_{yy} + \hat{v}^0_{xx}, \tag{4.2b}$$

$$\hat{u}^0_x + \hat{v}^0_y = 0, \tag{4.2c}$$

$$0 = \hat{u}^0_y(x, 1) = \hat{v}^0(x, 1) \quad \text{for} \quad x > 0, \tag{4.2d}$$

$$0 = \hat{u}^0(x, 1) = \hat{v}^0(x, 1) \quad \text{for} \quad x < 0, \tag{4.2e}$$

$$0 = \hat{u}^0(x, 0) = \hat{v}^0(x, 0) \quad \text{for all} \quad x, \tag{4.2f}$$

$$\hat{u}^0 \to 2(y - y^2), \quad \hat{v}^0 \to 0 \quad \text{as} \quad x \to -\infty, \quad 0 < y < 1. \tag{4.2g}$$

This problem is formidable, even though the domain is rectangular. It should be possible to solve it approximately when inertial terms are negligible, as is likely when R is small (highly viscous liquid). With the acceleration terms removed the problem is linear, and when a stream function is introduced the field equations collapse into the biharmonic equation, which can be solved on the rectangular strip by a Fourier-transform method and the Wiener–Hopf technique. Richardson (1967, 1970) employed this solution procedure and Ruschak (1974) applied it to a similar problem.

Because the solution to (4.2) does not satisfy the downstream boundary conditions we rescale (2.1) and consider a second set of asymptotic expansions. The interval over which h is nearly 1 should expand downstream as $w \to 0$, and we anticipate that the new x co-ordinate will have the form $\tilde{x} = w^n x$, with $n > 0$. The rescaling can proceed in the manner sketched in the preceding section. The required second set of variables is

$$\left.\begin{aligned} \tilde{x} &= w^{\frac{2}{3}}x, \quad \tilde{y} = y, \quad \tilde{h} = h, \\ \tilde{u} &= u, \quad \tilde{v} = v/w^{\frac{2}{3}}, \quad \tilde{p} = w^{\frac{2}{3}}p \end{aligned}\right\} \tag{4.3}$$

and the problem becomes

$$Rw^{\frac{2}{3}}[\tilde{u}\tilde{u}_{\tilde{x}}+\tilde{v}\tilde{u}_{\tilde{y}}] = -d^{-3}\tilde{p}_{\tilde{x}}+\tilde{u}_{\tilde{y}\tilde{y}}+w^{\frac{4}{3}}\tilde{u}_{\tilde{x}\tilde{x}}+d^{-3}, \tag{4.4a}$$

$$Rw^{2}[\tilde{u}\tilde{v}_{\tilde{x}}+\tilde{v}\tilde{v}_{\tilde{y}}] = -d^{-3}\tilde{p}_{\tilde{y}}+w^{\frac{4}{3}}[\tilde{v}_{\tilde{y}\tilde{y}}+w^{\frac{4}{3}}\tilde{v}_{\tilde{x}\tilde{x}}], \tag{4.4b}$$

$$\tilde{u}_{\tilde{x}}+\tilde{v}_{\tilde{y}} = 0, \tag{4.4c}$$

$$\left.\begin{aligned} -\tilde{p} &= d^{3}w^{\frac{4}{3}}[2\tilde{u}_{\tilde{x}}+\tilde{h}_{\tilde{x}}(\tilde{u}_{\tilde{y}}+w^{\frac{4}{3}}\tilde{v}_{\tilde{x}})]+\tilde{h}_{\tilde{x}\tilde{x}}[1+w^{\frac{4}{3}}\tilde{h}_{\tilde{x}}^{2}]^{-\frac{3}{2}} \quad (4.4d)\\ 0 &= [\tilde{u}_{\tilde{y}}+w^{\frac{4}{3}}\tilde{v}_{\tilde{x}}][1-w^{\frac{4}{3}}\tilde{h}_{\tilde{x}}^{2}]-4w^{\frac{4}{3}}\tilde{h}_{\tilde{x}}\tilde{u}_{\tilde{x}} \quad (4.4e)\end{aligned}\right\} \quad \text{on} \quad \tilde{y} = \tilde{h}(\tilde{x}),$$

$$\tilde{v} = \tilde{h}_{\tilde{x}}\,\tilde{u}, \tag{4.4f}$$

$$\tilde{u} = \tilde{v} = 0 \quad \text{on} \quad \tilde{y} = 0, \tag{4.4g}$$

$$\tilde{u}\to d^{-3}(d\tilde{y}-\tfrac{1}{2}\tilde{y}^{2}), \quad \tilde{v}\to 0, \quad \tilde{h}\to d \quad \text{as} \quad \tilde{x}\to\infty, \quad 0<\tilde{y}<d. \tag{4.4h}$$

The nearly rectilinear nature of the flow downstream of the slot is reflected in these equations. In the limit $w\to 0$ a much simpler problem, almost identical to (3.4), results. With a superscript zero denoting the limit of a function as $w\to 0$, we obtain [cf. (3.5)]

$$\tilde{p}^{0}_{\tilde{x}} = 1-d^{3}/(\tilde{h}^{0})^{3}, \tag{4.5a}$$

$$\tilde{u}^{0} = (\tilde{h}^{0})^{-1}(\tilde{y}/\tilde{h}^{0})[1-\tfrac{1}{2}(\tilde{y}/\tilde{h}^{0})], \tag{4.5b}$$

$$\tilde{v}^{0} = \tilde{h}^{0}_{\tilde{x}}(\tilde{y}/\tilde{h}^{0})\,\tilde{u}^{0}. \tag{4.5c}$$

Furthermore the differential equation and boundary conditions which determine the limit of the meniscus profile are

$$\left.\begin{aligned} \tilde{h}^{0}_{\tilde{x}\tilde{x}\tilde{x}} &= d^{3}(\tilde{h}^{0})^{-3}-1,\\ \tilde{h}^{0}(0) = 1; \quad \tilde{h}^{0} &\to d \quad \text{as} \quad \tilde{x}\to\infty,\end{aligned}\right\} \tag{4.6}$$

and when d is less than 1 the solution to (4.6) can be expressed in terms of the function $g(r)$ defined in (3.6):

$$\tilde{h}^{0}(\tilde{x}) = dg[\tilde{x}/d^{\frac{1}{3}}+g^{-1}(1/d)]. \tag{4.7}$$

Thus the flow downstream from the slot is nearly rectilinear and can be described using the lubrication approximation when w is small. The vertical component of velocity follows a semi-parabolic distribution given by (4.5). The pressure gradient is responsible for the departure from strictly rectilinear flow, and the slight curvature of the meniscus accommodates the pressure variation along the film. Across the film the pressure is uniform although it is not atmospheric.

It is possible to obtain the second terms in the downstream asymptotic expansions. Interestingly, the governing differential equations are the boundary-layer equations, with a pressure gradient imposed by the action of surface tension in the curved meniscus. The problem depends on $\tilde{h}^{0}(\tilde{x})$ and its derivatives, and so numerical solution is required except in the special case $d\sim 1$.

In the preceding section we found that the low-flow-rate limit of the free surface near the slot exit is a static meniscus originating at the slot edge and contacting the wall tangentially and with zero curvature. To show how this singular limit can arise we now examine low-flow-rate expansions of the high-surface-tension result downstream of the slot exit. The expression for the thickness of the liquid film (4.7) can be written as

$$\tilde{h}^{0}(\tilde{x}) = dg[d^{-\frac{1}{3}}(\tilde{x}-6^{\frac{1}{3}})+c_{0}-26^{-\frac{1}{3}}I_{1}\,d^{\frac{1}{3}}+26^{-\frac{2}{3}}I_{2}\,d^{\frac{2}{3}}+\ldots]. \tag{4.8}$$

When $\tilde{x}$ is held fixed, $0 < \tilde{x} < 6^{\frac{1}{3}}$, (4.8) leads to the expansion

$$\tilde{h}^0(\tilde{x}) \sim -\tfrac{1}{6}(\tilde{x}-6^{\frac{1}{3}})^3 + I_1[(\tilde{x}-6^{\frac{1}{3}}) + 6^{-\frac{1}{3}}(\tilde{x}-6^{\frac{1}{3}})^2]\, d^{\frac{2}{3}} + \dots . \tag{4.9}$$

The leading term here is identical to the leading term of (3.18), which is an expansion for small d and small w, and it indeed represents a static meniscus which has no curvature at the point where it is tangential to the wall. Furthermore there is no term of order $d^{\frac{1}{3}}$, in line with the remarks just after (3.20). Finally it is evident from (4.8) that the limit of $\tilde{h}^0(\tilde{x})$ is zero when $\tilde{x} > 6^{\frac{1}{3}}$.

The form of a boundary-layer variable is apparent in (4.8). We let

$$z = (\tilde{x}-6^{\frac{1}{3}})/d^{\frac{1}{3}} \tag{4.10}$$

and with z fixed again expand (4.8). With z fixed $\tilde{x}$ approaches the point of tangency of the static meniscus with the wall:

$$\tilde{h}^0(\tilde{x})/d \sim g(z+c_0) - 26^{-\frac{1}{3}} I_1\, g'(z+c_0)\, d^{\frac{1}{3}} + 26^{-\frac{2}{3}}[I_2\, g'(z+c_0) + I_1^2\, g''(z+c_0)]\, d^{\frac{2}{3}} + \dots . \tag{4.11}$$

This second expansion corresponds to (3.21). This short exercise provides added insight into the singular – and perhaps surprising – nature of the small-flow-rate limiting case, in which a viscous transition region or boundary layer emerges.

5. Concluding remark

The analysis of low flow rates, corresponding to thin films, has some kinship with analyses of dip coating, in which a film is deposited on a sheet or ribbon that is drawn out of a liquid bath. The interested reader may wish to compare our development with the classical investigation of dip coating by Landau & Levich (1942). There is a related analysis of the flow of liquid around long bubbles in tubes by Bretherton (1961). Neither of these studies employs the method of matched expansions. Both are correct as far as they go, but they are not complete. In particular they overlook the use of a boundary-layer co-ordinate pivoted about the locus where the limiting static meniscus would contact the wall. Consequently their matching procedures are *ad hoc*, and they do not make clear how higher-order effects could be calculated. The approach we have taken can be carried over to these older problems.

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